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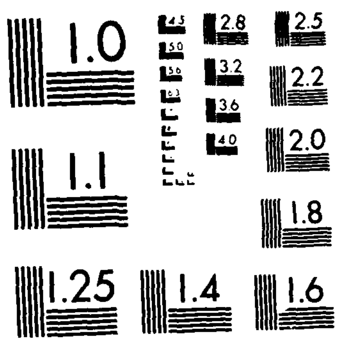

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Technical Report "Eigenvectors of Graphs"

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Abstract

In this work we considered the rows of an eigenmatrix of a distance-regular graph as coordinates of points in space. Geometric and algebraic properties and relations for these points are found that correspond to graph-theoretic properties of the corresponding vertices. Some properties and relations considered are automorphisms, proximity, linear dependence and independence, and facets of the convex hull.

1. Introduction

Let G be a graph with vertex set $\{1,2,\ldots,n\}$, A its adjacency matrix, $\alpha_1 > \alpha_2 > \ldots > \alpha_s$ its distinct eigenvalues with multiplicities $m_1, m_2, \ldots, m_s$. For each eigenvalue α, there is an $n \times m$ ($m = \text{mult}(\alpha)$) eigenmatrix Z satisfying $AZ = \alpha Z$, $Z^T Z = I$.

In this report we summarize research into properties of eigenmatrices of graphs that are distance-regular. For definitions, see Biggs (1974).

2. Automorphism group

Godsil (1978) and Babai (1978) proved slightly varying versions of this theorem. (Also see Cvetkovic et al., 1980, sec. 5.2).

Theorem. Aut(G), the automorphism group of G, is isomorphic to a subgroup of the direct sum of the orthogonal groups of degrees m_1, $m_2, \ldots, m_s$.

By restricting the class of graphs, we are able to get a much sharper result. (Powers 1987c)

Theorem 1. Let G be distance-regular. Then aut(G) is isomorphic to the group $\text{orth}(Z) = \{R\colon ZR = PZ;\ P \text{ a permutation}\}$, where Z is an eigenmatrix associated with the second eigenvalue of A.

It can be shown that any matrix R in orth(Z) is indeed orthogonal, so that a corollary is: aut(G) is isomorphic to a subgroup of the orthogonal group of degree m_2. We conjecture that this representation of aut(G) is irreducible for distance-regular graphs. Incidentally, the definition of orth(Z) was based on Coxeter's definition of a symmetry group of a "star." (Coxeter 1973, p. 253)

The proof of the preceeding theorem and several that follow depend on this lemma, which uses some of the ideas of Powers (1987a,b).

Lemma. Let G be distance regular, Z an eigenmatrix associated with $\alpha = \alpha_2$. Then (1) the projector $L = ZZ^T$ has the decomposition

$$L = y_0 A_0 + y_1 A_1 + y_2 A_2 + \ldots + y_d A_d,$$

where A_k has a 1 in the i,j-position if dist(i,j) = k or a 0 otherwise, and d is the diameter of G; (2) the coefficients satisfy $y_0 > y_1 > y_2 \geq \cdots \geq y_d$.

3. Rows of an eigenmatrix

Let Z be an n x m eigenmatrix associated with the eigenvalue α of A. We consider the rows of

$$Z = \begin{pmatrix} W_1^T \\ \cdot \\ \cdot \\ W_n^T \end{pmatrix}$$

to be (coordinates of) points in euclidean m-space. There is a natural correspondence between the row W_i^T of Z and the vertex i of G. We attempt to find algebraic and geometric relations among the rows that correspond to relations among the corresponding vertices. Three theorems in this vein follow (Powers 1987c). All assume that the graph G is distance-regular and that Z is an eigenmatrix corresponding to α, the second eigenvalue, but some can be generalized to other cases.

Theorem 2. The rows of Z corresponding to a vertex and all its neighbors are linearly dependent.

Theorem 3. The rows of Z corresponding to the vertices of a clique are linearly independent.

Theorem 4. For each i, the rows W_j^T for which the euclidean distance $\|W_i^T - W_j^T\|$ ($j \neq i$) is minimized correspond to the vertices adjacent to vertex i.

From Theorem 3 it follows that the multiplicity of α_2 is at least as large as the order of a clique. (Compare Terwilliger 1982.) The proof of Theorem 2 shows that a row W_i^T lies on the same side of the origin as the rows W_j^T corresponding to neighbors of i.

In this context, we can also present a result of Powers (1987a).

Theorem 5. Let $u \neq 0$ be an arbitrary point in m-dimensional euclidean space, and define

$$U = \{i: e_i^T Zu \geq 0\}.$$

Then the subgraph of G induced by vertices in U is connected.

4. Polytopes

Another way to organize the geometric information about the rows of Z is by examining the convex polytope $P(\alpha)$ that is the convex hull of the rows of Z (Godsil 1978). For a distance-regular graph G, α the second eigenvalue, each row of Z is an extreme point of $P(\alpha)$. In Powers (1987c) we obtain the following theorems about polytopes.

Theorem 6. If G contains a clique on m vertices ($m = \text{mult}(\alpha)$) then the corresponding rows of Z are the extreme points of a facet of $P(\alpha)$ and that facet is a simplex.

Theorem 7. The rows of Z corresponding to a vertex and all its neighbors are not among the extreme points of any one facet of $P(\alpha)$.

Further theorems concerning the polytope $P(\alpha)$ are reported in Powers (1987c). In Licata and Powers (1987), several important examples were studied. The following theorem summarizes many of the results.

Theorem 8. If G is the skeleton of a polytope P from the list below, then the polytope $P(\alpha)$ associated with the second eigenvalue is geometrically similar to P.

(a) n-gons ($n \geq 3$)

(b) Platonic solids

(c) simplexes

(d) cross-polytopes

(e) generalized cubes.

We conjecture that all regular (and many semi-regular) polytopes belong on the list, and we have substantial experimental evidence in favor of the conjecture.

A separate study was made of the polytope P(1) for the Petersen graph (Powers 1986). The result was a complete description of an interesting case, but no very useful insight. The extreme points of the facets correspond to the vertices of a 5-cycle or a 6-cycle in the graph.

5. Further work

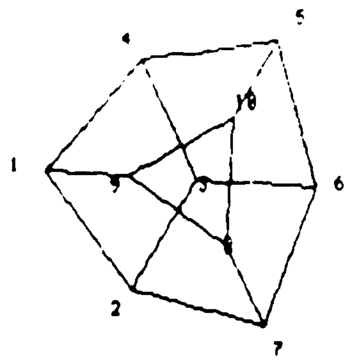
Geometric and algebraic properties of rows of eigenmatrices appear to offer opportunities for further development. Some projected areas of stuoy

(1) Study of rows of eigenmatrices associated with eigenvalues other than the second, in distance-regular graphs.

(2) Study the class of distance-regular graphs in which each vertex is at maximum distance from exactly one other. Some additional properties for this class were found in Powers (1987c). Many important graphs fit in this category, including most of those mentioned in Theorem 8.

(3) Relax the restriction to distance-regular graphs. Direct computation shows that many graphs that are not distance-regular share the properties mentioned in Theorems 3,4,6 and 8 for instance. H. Sachs (private communication) suggests that the property of Theorem 8 may provide a canonical representation for some polytopes.

(4) When all rows of Z have the same euclidean norm (e.g. for vertex-transitive and distance-regular graphs) the polytope mentioned in section 4 is useful. However, when this is not the case, the sets of vertices corresponding to rows of the same norm are fixed by the automorphism group of G. The figure below uses rows of a 10 x 2 eigenmatrix for coordinates of vertices and clearly illustrates this idea. Further exploitation may lead to some useful information about the automorphisms of G.



EIGENVALUE: 1.2470

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